Chapter 1

Introduction: From Science to Computation

1.1 From Art through Science to Computation

It will do very nicely to start with a description of what an engineer does, before we proceed to define briefly how and where structural mechanics plays a crucial role in the life of the engineer. The structural engineer's appointed office in society is to enclose space for activity and living, and sometimes does so giving time as well - the ship builder, the railway engineer and the aerospace engineer enable travel in enclosed spaces that provide safety with speed in travel. He did this first, by imitating the structural forms already present in Nature.

From Art imitating Life, one is led to codification of the accumulated wisdom as science - the laws of mechanics, elasticity, theories of the various structural elements like beams, plates and shells etc. From Archimedes' use of the Principle of Virtual Work to derive the law of the lever, through Galileo and Hooke to Euler, Lagrange, Love, Kirchhoff, Rayleigh, etc. we see the theoretical and mathematical foundations being laid. These where then copiously used by engineers to fabricate structural forms for civil and military functions. Solid and Structural Mechanics is therefore the scientific basis for the design, testing, evaluation and certification of structural forms made from material bodies to ensure proper function, safety, reliability and efficiency.

Today, analytical methods of solution, which are too restricted in application, have been replaced by computational schemes ideal for implementation on the digital computer. By far the most popular method in Computational Structural Mechanics is that called the Finite Element Method. People who use computational devices (hardware and software) with hardly any reference to experiment or theory now perform design and Analysis. From advertising claims made by major fem and cad software vendors (5000 sites or installations of one package, 32,000 seats of another industry standard vendor, etc.); it is possible to estimate the number of fem computationalists or analysts as lying between a hundred thousand and two hundred thousand. It is to them that this book is addressed.

1.2 Structural Mechanics

A structure is "any assemblage of materials which is intended to sustain loads". Every plant and animal, every artifact made by man or beast has to sustain greater or less forces without breaking. The examination of how the laws of nature operate in structural form, the theory of structural form, is what we shall call the field of structural or solid mechanics. Thus, the body of knowledge related to construction of structural form is collected, collated, refined, tested and verified to emerge as a science. We can therefore think of solid and structural mechanics as the scientific basis for the design, testing, evaluation and certification of structural forms made from material bodies to ensure proper function, safety, reliability and efficiency.

To understand the success of structural mechanics, one must understand its place in the larger canvas of classical mechanics and mathematical analysis, especially the triumph of the infinitesimal calculus of continuum behavior in physics. The development and application of complex mathematical tools led to the growth of the branch of mathematical physics. This therefore encouraged the study of the properties of elastic materials and of elasticity - the description of deformation and internal forces in an elastic body under the action of external forces using the same mathematical equipment that was been used in other classical branches of physics. Thus, from great mathematicians such as Cauchy, Navier, Poisson, Lagrange, Euler, Sophie Germain, came the formulation of basic governing differential equations which originated from the application of the infinitesimal calculus to the behavior of structural bodies. The study is thus complete only if the solutions to such equations could be found. For nearly a century, these solutions were made by analytical techniques however, these were possible only for a very few situations where by clever conspiracy, the loads and geometries were so simplified that the problem became tractable. However, by ingenuity, the engineer could use this limited library of simple solutions to construct meaningful pictures of more complicated situations.

It is the role of the structural designer to ensure that the artifacts he designs serve the desired functions with maximum efficiency, but putting only as much flesh as is needed on the bare skeleton. Often, this is measured in terms of economy of cost and/or weight. Thus for vehicles,low weight is of the essence, as it relates directly to speed of movement and cost of operation. The efficiency of such a structure will depend on how every bit of material that is used in components and joints is put to maximum stress without failing. Strength is therefore the primary driver of design. Other design drivers are stiffness (parts of the structures must not have excessive deflections), buckling (members should not deflect catastrophically under certain loading conditions), etc. It is not possible to do this check for structural integrity without having sophisticated tools for analysis. FEM packages are therefore of crucial relevance here. In fact, the modern trend is to integrate fem analysis tools with solid modeling and CAD/CAM software in a single seamless chain or cycle all the way from concept and design to preparation of tooling instructions for manufacture using numerically controlled machines.

1.3 From analytical solutions to computational procedures

1.3.1 Introduction

We have seen earlier in this chapter how a body of knowledge that governs the behavior of materials, solids and structures came to exist. Gifted mathematicians and engineers were able to formulate precise laws that governed the behavior of such systems and could apply these laws through mathematical models that described the behavior accurately. As these mathematical models had to take into account the continuum nature of the structural bodies, the description often resulted in what are called differential equations. Depending on the sophistication of the description, these differential equations could be very complex. In a few cases, one could simplify the behavior to gross relationships - for a bar, or a spring, it was possible to replace the differential equation description with a simple relation between forces and displacements; we shall in fact see that this is a very simple and direct act of the discretization process that is the very essence of the finite element process. For a more general continuum structure, such discretizations or simplifications were not obvious at first. Therefore, there was no recourse

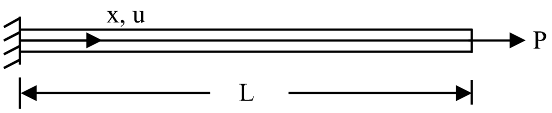

Fig. 1.1 A simple bar problem under axial load

except to find by mathematical and analytical techniques, viable solutions to the governing differential equations. This can become very formidable in cases of complex geometry, loading and material behavior, and often were intractable; however with ingenuity, solutions could be worked out for a useful variety of structural forms. Thus a vast body of solutions to the elastic, plastic, viscoelastic behavior of bars, beams, plates and shells has been built over the last two centuries.

It was recognized quite early that where analytical techniques fail or were difficult to implement, approximate techniques could be devised to compute the answers. In fact, much before algebra and analysis arrived, in contexts more general than solid or structural mechanics, simple approximation and computational schemes were used to solve engineering and scientific problems. In this chapter, we shall review what we mean by an analytical solution for a very simple problem and then proceed to examine some computational solutions. This will reveal to us how the discretization process is logically set up for such a problem.

1.3.2 The simple bar problem

Perhaps the earliest application of the discretization technique as it appeared in civil engineering practice was to the bar problem. The bar is a prismatic one-dimensional structure which can resist only axial loads, in tension or in compression, and cannot take any bending loads. Figure 1.1 shows its simplest configuration - a cantilever bar. This is a statically determinate problem, meaning that one can obtain a complete solution, displacements as well as strains and stresses from considerations of equilibrium alone. We shall now compute the analytical solution to the problem depicted in Fig.1.1 using very elementary engineering mathematics.

1.3.2.1 Analytical solution

The problem is categorized as a boundary value problem. We presume here that for this problem, the reader is able to understand that the governing differential equation describing the situation is

$$EA\ u_{,xx} = 0 \qquad (1.1)$$

where E is the Young’s modulus, A is the area of cross-section, $u(x)$ is the function describing the variation of displacement of a point, and $_{,x}$ denotes differentiation with respect to coordinate x. This is the equation of equilibrium describing the rate at which axial force varies along the length of the bar. In general, governing differential equations belong to a category called partial differential equations but here we have a simpler form known as an ordinary differential equation. Equation (1.1) is further classified as a field equation or condition as one must find a solution to the variable u which must satisfy the equation over the entire field or domain, in this case, for x ranging from 0 to L. A solution is obviously

$$u = ax + b \tag{1.2}$$

where a and b are as yet undetermined constants. To complete the solution, i.e. to determine a and b in a unique sense for the boundary (support) and loading conditions shown in Fig. 1.1, we must now introduce what are called the boundary conditions. Two boundary conditions are needed here and we state them as

$$u = 0 \text{ at } x = 0 \tag{1.3a}$$
$$EA\, u,_x = P \text{ at } x = L \tag{1.3b}$$

The first relates to the fact that the bar is fixed at the left end and the second denotes that a force P is applied at the free end. Taking these two conditions into account, we can show that the following description completely takes stock of the situation:

$$u(x) = Px/EA \tag{1.4a}$$
$$\varepsilon(x) = P/EA \tag{1.4b}$$
$$\sigma(x)= E\varepsilon(x)= P/A \tag{1.4c}$$
$$P(x) = A\sigma(x) = P \tag{1.4d}$$

where $\varepsilon(x)$, $\sigma(x)$ and $P(x)$ are the strain, stress and axial force (stress resultants) along the length of the bar. These are the principal quantities that an engineer is interested in while performing stress analysis to check the integrity of any proposed structural design.

1.3.2.2 Approximate solutions

It is not always possible to determine exact analytical solutions to most engineering problems. We saw in the example above that an analytical solution provides a unique mathematical expression describing in complete detail the value of the field variable at every location in the body. From this expression, other expressions can be derived which describe further quantities of practical interest, e.g. strains, stresses, stress resultants, etc.

In most problems where analytical solutions appear infeasible, it is necessary to resort to some approximate or numerical methods to obtain these values of practical interest. One obvious method at this stage is to start with the equations we have already derived, namely Equations (1.1) and (1.3). This can be achieved using a technique known as the finite difference method. We shall briefly describe this next.

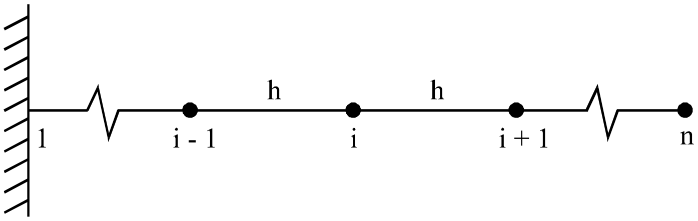

Fig. 1.2 Grid for finite difference scheme

Other approximate solutions are based on what are called functional descriptions or variational descriptions of the problem. These describe the problem at a level which is more fundamental in the laws of physics than the more specialized descriptions in terms of governing differential equations as seen in Equations (1.1) and (1.3) above. We deal directly with energy or work quantities and discretization or approximation is applied at that level. If we can appreciate that the governing differential equations were derived analytically from such energy or work principles using what is called a variational or virtual work approach, then it would become easier to understand that an approximation applied directly to the energy or work quantities and a variation carried out subsequently on these terms will preserve both the physics as well as the approximating or numerical solution in one single process. This will be the basis for the finite element method as we shall see in subsequent subsections.

1.3.2.3 Finite difference approximations

For a problem where the differential equations are already known, the technique known as the finite difference method can be used to obtain an approximate or numerical solution. In fact, before the finite element method was established, it was this method which was used to obtain solutions for very complicated problems in structural mechanics, fluid dynamics and heat transfer.

Figure 1.2 shows a very simple uniformly spaced mesh or grid of nodal points that can be used to discretize the problem described by Equations (1.1) and (1.3). The larger the number of nodal points used (i.e. the smaller the grid spacing h), the greater will be the accuracy involved. The field variable is now taken to be known (or very strictly, unknown, at this stage of the computation) at these nodal points. Thus, $u_1, \ldots, u_i, \ldots, u_n$, are the unknown variables or degrees of freedom. The next step is to rewrite the governing differential equations and boundary conditions in finite difference form. We see that the finite difference forms for $u_{,x}$ and $u_{,xx}$ are required. It is easy to show (again, details are omitted, as these can be found in most books on numerical analysis or fem) that at a grid point i.

$$u_{,x} = (u_{i+1} - u_i)/h \tag{1.5a}$$

$$u_{,xx} = (u_{i+1} - 2u_i + u_{i-1})/h^2 \tag{1.5b}$$

We now attempt a solution in which four points are used in the finite difference grid. The governing differential equations and boundary conditions are replaced by the following discrete set of linear, algebraic equations:

$$u_1 = 0 \tag{1.6a}$$

$$u_3 - 2u_2 + u_1 = 0 \tag{1.6b}$$

$$u_4 - 2u_3 + u_2 = 0 \tag{1.6c}$$

$$u_4 - u_3 = Ph/EA \tag{1.6d}$$

where $h=L/3$ and Equations (1.6a) and (1.6d) represent the boundary conditions at $x = 0$ and L respectively. The reader can easily work out that the solution obtained from this set of simultaneous algebraic equations is, $u_2 = PL/3EA$, $u_3 = 2PL/3EA$ and $u_4 = PL/EA$. Comparison with Equation (1.4a) will confirm that we have obtained the exact answers at the grid points. Other quantities of interest like strain, stress, etc. can be computed from the grid point values using the finite difference forms of these quantities.

The above illustrates the solution to a very simple one-dimensional problem for which the finite difference procedure yielded the exact answer. Generalizations to two and three dimensions for more complex continuous problems can be made, especially if the meshes are of regular form and the boundary lines coincide with lines in such regular grids. For more complex shapes, the finite difference approach becomes difficult to use. It is in such applications that the finite element method proves to be more versatile than the finite difference scheme.

1.3.2.4 Variational formulation

We began this chapter with an analytical solution to the differential equations governing the problem. We did not ask how these equations originated. There are two main ways in which it can be done. The first, and the one that is most often introduced at the earliest stage of the study of structural mechanics, is to formulate the equations of equilibrium in terms of force or stress quantities. Then, depending on considerations of static determinacy, make the problem determinate by introducing stress-strain and strain-displacement relations until the final, solvable, set of governing equations is obtained.

The second method is one that originates from a more fundamental statement of the principles involved. It recognizes that one of the most basic and elegant principles known to man is the law of least action, or minimum potential energy, or virtual work. It is a statement that Nature always chooses the path of minimum economy. Thus, the law of least action or minimum total potential is as axiomatic as the basic laws of equilibrium are. One can start with one axiom and derive the other or vice-versa for all of mechanics or most of classical physics. In dynamics, the equivalent principle is known as Hamilton's principle.

In structural mechanics, we start by measuring the total energy stored in a structural system in the form of elastic strain energy and potential due to external loads. We then derive the position of equilibrium as that state where this energy is an extremum (usually a minimum if this is a stable state of equilibrium). This statement in terms of minimum energy is strictly true only for what are called conservative systems. For non-conservative systems, a more general statement known as the principle of virtual work would apply. From these brief general philosophical reflections we shall now proceed to the special case at hand to see how the energy principle applies.

The total energy is stated in the form of a functional. In fact, most of the problems dealt with in structural mechanics can be stated in a variational form as the search for a minimal or stationary point of a functional. The functional, Π, is an integral function of functions which are defined over the domain of the structure. Thus, for our simple bar problem, we must define terms such as the strain energy, U, and the potential due to the load, V, in terms of the field variable $u(x)$. This can be written as

$$U = \int_0^L 1/2\, EA\ u^2,_x\ dx \tag{1.7a}$$

$$V = P\ u_{x=L} \tag{1.7b}$$

Then,

$$\Pi(\upsilon(x)) = \int_0^L 1/2\, EA\ u^2,_x\ dx - P\, u_{x=L} \tag{1.8}$$

Using standard variational procedures from the calculus of variations, we can show that the extremum or stationary value of the functional is obtained as

$$\delta\Pi = 0 \tag{1.9}$$

If this variation is carried out, after integrating by parts and by regrouping terms, we will obtain the same equation of equilibrium and boundary conditions that appeared in Equations (1.1) and (1.3) above. Our interest is now not to show what can be done with these differential equations but to demonstrate that the approximation or discretization operation can be implemented at the stage where the total potential or functional is defined.

1.3.2.5 Functional approximation

There are several ways in which an approximation can be applied at the level of energy, virtual work or weighted residual statements, e.g. the Rayleigh-Ritz (R-R) procedure, the Galerkin procedure, the least-squares procedure, etc. We first choose a set of independent functions which satisfy the geometric or essential (also called kinematic) boundary conditions of the problem. These functions are called admissible functions. In this case, the condition specified in Equation (1.3a) is the geometric boundary condition. The other condition, Equation (1.3b) is called a nonessential or natural or force boundary condition. The admissible functions need not satisfy the natural boundary conditions (in fact, it can be shown that if they are made to do so, there can even be a deterioration in performance). The approximate admissible configuration for the field variable, say $\overline{u}(x)$, is obtained by a linear combination of these functions using unknown coefficients or parameters, also known as degrees of freedom or generalized coordinates. These unknown constants are then determined to satisfy the extremum or stationary statement. Methods like the Rayleigh-Ritz procedure (the one going to be used now), the Galerkin method, collocation methods, least square methods, etc. all proceed with approximating fields chosen in this way.

For our problem, let us choose only one simple linear function, using a polynomial form for the purpose. The advantages of using polynomial functions will become clear later in the book. Thus, only one constant is required, i.e.

$$\bar{u}(x) = a_1 x \tag{1.10}$$

Note that this configuration is an admissible one satisfying the geometric condition $\bar{u}(0) = 0$. By substituting in Equation (1.8) and carrying out the variation prescribed in Equation (1.9), we are trying to determine the value of a_1 which provides the best functional approximation to the variational problem. It is left to the reader to show that the approximate solution obtained is

$$\bar{u}(x) = Px/EA \tag{1.11a}$$

$$\bar{\varepsilon}(x) = P/EA \tag{1.11b}$$

$$\bar{\sigma}(x) = E\varepsilon(x) = P/A \tag{1.11c}$$

$$\bar{P}(x) = A\sigma(x) = P \tag{1.11d}$$

Thus, for this problem, the approximate solution coincides with the exact solution. Obviously, we cannot draw any conclusions here as to the errors involved in approximation by the R-R procedure. In more complicated problems, approximate solutions will be built up from several constants and admissible functions and convergence to the exact solution will depend on the number of terms used and the type of functions used.

The R-R procedure, when applied in a piecewise manner, over the element domains that constitute the entire structure, becomes the finite element method. Thus, an understanding of how the R-R method works will be crucial to our understanding of the finite element method.

1.3.2.6 Finite element approximation

We shall now describe an approximate solution by the finite element method. We shall use a form known as the displacement-type formulation (also called the stiffness formulation). The simplest element known for our purpose is a two noded line element such as shown in Fig. 1.3. This element is also known as the bar, truss or rod element. It has two nodes at which nodal displacements u_1 and u_2 are prescribed. These are also called the nodal degrees of freedom. Also indicated in Fig. 1.3 are the forces P_1 and P_2 acting at these nodes.

Thus, any one-dimensional structure can be replaced by contiguous elements placed from end to end. The only information communicated from element to element is that stored as nodal degrees of freedom and the nodal forces. We can now see how this piecewise Ritz approximation is performed.

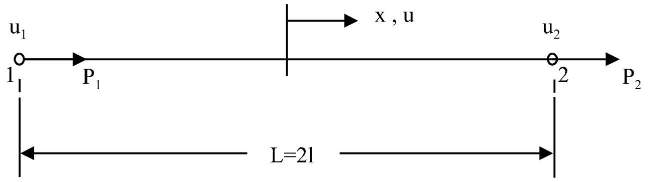

Fig. 1.3 A two-node bar element

We first derive an expression for the approximate representation of the field variable, $\overline{u}(x)$, within the region of the element by relating it to the nodal degrees of freedom. It can be shown that,

$$\overline{u}(x) = u_1 N_1 + u_2 N_2 \tag{1.12}$$

where $N_1 = (1-\xi)/2$ and $N_2 = (1+\xi)/2$, where $\xi = x/l$. N_1 and N_2 are called the shape functions. Note that this form is exactly the same as the linear function used in our R-R approximation earlier, Equation (1.10), with generalized coordinates a_i being now replaced by nodal quantities u_i. It is possible therefore to compute the energy stored in a beam element as

$$\{u_1\ u_2\} \frac{EA}{2l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{1.13}$$

and the potential due to the applied loads as

$$\{u_1\ u_2\} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \tag{1.14}$$

Thus, if a variation is taken over u_1 and u_2, we can write the equation of equilibrium as

$$\frac{EA}{2l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \tag{1.15}$$

This matrix representation is typical of the way finite element equations are assembled and manipulated automatically on the digital computer. We shall now attempt to solve our bar problem using a finite element model comprising just one element. Node 1 is therefore placed at the fixed end and node 2 coincides with the free end where the load $P_2 = P$ is applied. The fixity condition at node 1 is simply $u_1 = 0$ and P_1 indicates the reaction at this end. It is very simple to show from these that $u_2 = PL/EA$. We can see that the exact solution has been obtained.

What we have seen above is one of the simplest demonstrations of the finite element method that is possible. Generalizations of this approach to two and three dimensional problems permit complex plate, shell and three dimensional elasticity problems to be routinely handled by ready made software packages.

1.4 Concluding remarks

In this chapter, we have briefly investigated how classical analytical techniques, which are usually very limited in scope, can give way to computational methods of obtaining acceptable solutions. We have seen that the finite element method is one such approximate procedure. By using a very large number of elements, it is possible to obtain solutions of greater accuracy. Over the years, finite element theorists and analysts have produced a very large body of work that shows how such solutions are managed for a great variety of problems. This is in the first-order tradition of the method. The remaining part of this book hopes to address questions like: How good is the approximate solution? What kinds of situation affect these approximations in a trivial and in not-so-trivial ways? Since solutions depend very much on the number of elements used and the type of shape function patterns used within each element, we must ask, what patterns are the best to assume? All these questions are linked to one very fundamental question: In what manner does the finite element method actually compute the solution? Does it make its first approximation of the displacement field directly or on the stress field? The exercises and exposition in this book are aimed at throwing more light on these questions.